Theoretical Computer Science 125 (1994) 329–337
Elsevier

Note

Lyapunov operators to study the convergence of extremal automata

Eric Goles*

Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Casilla 170-correo 3, Santiago, Chile

Communicated by D. Perrin
Received June 1993

Abstract

Goles, E., Lyapunov operators to study the convergence of extremal automata, Theoretical Computer Science 125 (1994) 329–337.

In this work we associate discrete Lyapunov operators to a class of automata networks called extremal automata. These operators allow us to characterize the steady state and to give bounds for the transient time.

1. Introduction

Let $G=(V,E)$ be a finite, undirected, connected graph with values $x_i \in Q=\{q_1, \ldots, q_n\}$ assigned to each vertex. In this paper we shall study the parallel dynamics induced by the synchronous application of extremal rules (ER), i.e. each vertex value remains unchanged or it takes the maximum or the minimum between its neighbors. Particular cases of this class of discrete dynamical system were first proposed, in the framework of enhancement of digital images [5]. Convergence and transient time results have been developed in [1,4] where, for particular cases of ER; e.g., the forced move-stay rules (FES), it was proved that in steady state there exist only fixed points.

Correspondence to: E. Goles, Departo. de Ingeniería Matemática, Fac. Ciencias Físicas y Matemáticas, Universidad de Chile, Casilla 170 correo 3, Santiago, Chile. Email: egoles@uchcecvm.bitnet.

* Partially supported by FONDECYT-91-1211, DTI-Universidad de Chile and CEE.

SSDI 0304-3975(93)E0137-S

Furthermore, for the nearest-extremum rule (NE), a quadratic bound, $O(|V|^2)$, was obtained in [1], but no examples of effective $O(|V|^2)$ transient time were given.

In this paper we study convergence and transient bounds for some extremal rules. More precisely, we exhibit a simple ER, the max–min rule, which has an $O(|V|^2)$ transient time. Furthermore, there exists a Lyapunov functional (i.e. decreasing operator in the transient phase) which permits to give a $O(|V|^2)$ bound for the max–min rules. Both results prove that the order of the bound is reached.

On the other hand, we study, from the Lyapunov functional point of view, the NE class proposed in [1]. Concretely we exhibit a Lyapunov functional driving its synchronous dynamics which permits us to give an $O(|V|^2)$ transient bound as in [1].

2. Extremal rules

Let $V_i=\{j\in V/(i,j)\in E\}$, the function $f_i: Q^{|V_i|}\to Q$ is an extremal rule (ER) iff

$$f_i(x_j: j\in V_i)\in\{m_i, x_i, M_i\} \text{ and } f_i(x_j: j\in V_i)=x_i \quad \text{if } x_i\notin]m_i, M_i[,$$

i.e. vertex i remains unchanged or it changes only to extremum value in its neighbor. It is obvious that ER contains FES rules. In the FES case $x_i\in]m_i, M_i[$ implies $f_i(x_j: j\in V_i)\in\{m_i, M_i\}$. Analogous to the FES convergence theorem stated in [1], the synchronous update on ER converges to fixed points. To prove that, it is sufficient to consider the smallest value in a periodic orbit which changes.

For ER, the problem is to obtain bounds for the transient time, i.e. given a graph $G=(V,E)$ with ER update rules, one defines

$$\tau(G)=\begin{cases}\max\{t/t\geqslant 1,\ x(t)\neq x(t-1)\},\\ 0 \text{ otherwise.}\end{cases}$$

In this context, there exist classes of ER such that $\tau(G)=O(|V|^2)$.

Let us define the max–min rules (Mm) as follows: Let $\{I_M, I_m\}$ be a partition of V. Let $\{f_i\}$ be a set of ER and define

$$\forall i\in I_M,\quad f_i(x_j: j\in V_i)\neq x_i \Rightarrow f_i(x_j: j\in V_i)=M_i,$$

$$\forall i\in I_m,\quad f_i(x_j: j\in V_i)\neq x_i \Rightarrow f_i(x_j: j\in V_i)=m_i.$$

It is not difficult to see that one may always assume that, for Mm rules, the set of states is $Q=\{1,\dots,n\}$ where $n=|V|$. We have the following result.

Theorem 2.1. *Given $G=(V,E)$, the quantity $H(x(t))=\sum_{i\in I_m} x_i(t)-\sum_{i\in I_M} x_i(t)$ is a Lyapunov operator for the synchronous dynamics of Mm on graph G.*

Proof. Given $x(t+1)$, i.e. $x_i(t+1)=f_i(x_j(t): j\in V_i)$, it is obvious from the definition of Mm that $x_i(t+1)\leqslant x_i(t)$ for $i\in I_m$ and $x_i(t+1)\geqslant x_i(t)$ for $i\in I_M$, so if $x(t+1)\neq x(t)$, $\Delta H=H(x(t+1))-H(x(t))<0$. $\square$

Corollary 2.2. *For Mm, the transient time is bounded by* $\tau_{Mm}(G)\leqslant n(n-1)$.

Proof. It suffices to remark that

$$|I_m|-n|I_M|\leqslant H(x(t))\leqslant n|I_m|-|I_M| \quad \text{and} \quad x(t+1)\neq x(t) \ \Rightarrow\ |\Delta H|\geqslant 1. \qquad \square$$

We then have $\tau(G)=\mathrm{O}(|V|^2)$. Furthermore, we can exhibit the following theorem for Mm rules quadratic transients.

Theorem 2.3. *There exists* $G=(V,E)$ *such that, under Mm,* $\tau_{Mm}(G)=\mathrm{O}(|V|^2)$.

Proof. Let $G=(V,E)$ be such that $V=\{1,\dots,2n-1\}$ and the edges are defined as follows:

$$\forall i\in\{1,\dots,n-1\},\qquad \forall j\in\{n,\dots,n+i-1\},\qquad (i,j)\in E,$$

$$\forall i\in\{2,\dots,n-2\},\qquad (i-1,\ i),(i,i+1)\in E,$$

$$\forall i\in\{n+1,\dots,2n-2\},\qquad (i-1,\ i),(i,\ i+1)\in E.$$

Clearly, $|V|=2n-1$. A representation of G, for $|V|=9$ and $n=5$, is given in Fig. 1.

Consider the initial configuration

$$x_1(0)=1, \quad \text{and} \quad x_i(0)=2, \quad i\in\{2,\dots,n-1\},$$

$$x_{n+k}(0)=k+3, \quad k\in\{0,\dots,n-1\}.$$

A particular case is exhibited in Fig. 1.

Let $I_m=\{1,2,\dots,n-1\}$, $I_M=\{n,\dots,2n-1\}$ be the sets associated to Mm and the local rules.

For $i\in I_m$:

$$f_i(x_1,x_n,\dots,x_{n+i-1},x_i)=\begin{cases} m_i & \text{iff } x_n=\cdots=x_{n+i-1} \\ x_i & \text{otherwise.}\end{cases}$$

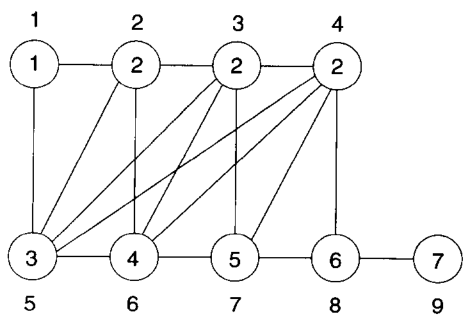

Fig. 1. The graph G with initial configuration $(1,2,2,2,3,4,5,6,7)$.

For $n+i-1 \in I_M$:

$$f_n(x_1, x_{n+1}, x_2, \ldots, x_{n-1}) = M_1 \quad \text{iff } x_1 = 1$$

$$f_{n+i-1}(x_i, x_{n+i-2}, x_{n+i}, x_{i+1}, \ldots, x_{n-1}) = \begin{cases} M_i & \text{iff } x_i = 1 \\ x_i & \text{otherwise} \end{cases}$$

$$f_{2n-1}(x_{2n-2}, x_{2n-1}) = M_i$$

where m_i and M_i are the local minimum and maximum, respectively.

Let us see the steps of the iteration:

$$\begin{array}{cccccc} 1 & 2 & 2 & \cdots & 2 & \\ 3 & 4 & 5 & \cdots & n+2 & n+2 \end{array} \rightarrow \begin{array}{cccccc} 1 & 2 & 2 & \cdots & 2 & \\ 4 & 4 & 5 & \cdots & n+2 & n+2 \end{array}$$

$$\cdots \rightarrow \begin{array}{ccccccc} & & \mathit{site}\ i & & & & \\ & & \downarrow & & & & \\ 1 & \cdots & 2 & 2 & \cdots & 2 & \\ n+i & & n+i & n+i+1 & \cdots & n+1 & n+2 \end{array}$$

$$\rightarrow \begin{array}{ccccccc} & & \mathit{site}\ i & & & & \\ & & \downarrow & & & & \\ 1 & \cdots & 1 & 2 & \cdots & 2 & \\ n+i & & n+i & n+i+1 & \cdots & n+1 & n+2 \end{array}$$

$$\rightarrow \begin{array}{ccccccc} & & \mathit{site}\ i & & & & \\ & & \downarrow & & & & \\ 1 & \cdots & 1 & 2 & \cdots & 2 & \\ n+i & & n+i+1 & n+i+1 & \cdots & n+1 & n+2 \end{array}$$

$$\cdots \rightarrow \begin{array}{ccccccc} & & & \mathit{site}\ i+1 & & & \\ & & & \downarrow & & & \\ 1 & \cdots & 1 & 2 & \cdots & 2 & \\ n+i+1 & \cdots & n+i+1 & n+i+1 & \cdots & n+1 & n+2 \end{array}$$

$$\cdots \rightarrow \begin{array}{cccccc} 1 & \cdots & \cdots & \cdots & 1 & \\ n+2 & \cdots & \cdots & \cdots & n+2 & n+2 \end{array} \qquad \text{fixed point}$$

It is obvious that the transient time for previous configuration is $\sum_{i=2}^{n} i = n(n+1)/2 - 1 = O(n^2)$. $\square$

3. Lyapunov functional for nearest extremum rules (NE)

Let us consider the NE rule proposed in [1], call $m_i = \min\{x_j\colon j \in V_i \cup \{i\}\}$, $M_i = \max\{x_j\colon j \in V_i \cup \{i\}\}$, where $V_i = \{j \in V\colon (i,j) \in E\}$ is the neighborhood of vertex $i \in V$.

The new value on vertex i is given by

$$f(x_j\colon j\in V_i)=\begin{cases} m_i & \text{if } x_i-m_i<M_i-x_i,\\ M_i & \text{if } x_i-m_i>M_i-x_i,\end{cases} \tag{3.1}$$

i.e. $f(x_j\colon j\in V_i)$ corresponds to the nearest extremum. Note that, without loss of generality, one may always assume that no value in set Q is the average of two others (see for instance [1]). Furthermore, the choice $q_i=(i+1)+1/(i+1)$, $i=1,\ldots,n$ verifies the previous condition and clearly, $\min_{i,j}(a_i-a_j)\geqslant 1$, $\max_{i,j}|a_i-a_j|<n-(1/2)$ so, in (3.1) there are no tie-cases.

The global dynamics updates all the sites synchronously, i.e. let $x(t)\in Q^{|V|}$ be the configuration at step t, then

$$x_i(t+1)=f(x_j(t)\colon j\in V_i),\quad 1\leqslant i\leqslant|V|=n.$$

It was proved in [1] that the iteration converges to fixed points in $O(n^2)$ steps. To achieve the proof, the authors introduce the notion of supreme jumps, which takes into account the maximum change of states between consecutive configurations during the dynamics history. Here, by using only the local maximum jump approach (i.e. between consecutive steps) and some of ideas developed in [1], we introduce an aditive local monotonous operator which is increasing during the transient phase of the dynamics; i.e. a discrete Lyapunov operator.

Let us recall the history approach developed in [1].

Lemma 3.1 (Goddard and Kleitman [1]). (a) *If the vertex i has a jump $\Delta=|x_i(t)-x_i(t-1)|$ at step t, then there exists a history of jumps of at least equal size in previous steps.*

(b) *If at step t_0 there exists a maximum jump Δ at vertex i then:*

(i) If $x_i(t_0+1)=M_i(t_0)$ $\Rightarrow$ $\{x_i(t)\}_{t\geqslant t_0}\nearrow$, $\{x_\alpha(t)\}_{t\geqslant t_0}\searrow$ *where $\alpha\in V_i$ realizes the minimum at step t_0; $x_\alpha(t_0)=m_i(t_0)$.*

(ii) *If $x_i(t_0+1)=m_i(t_0)$ $\Rightarrow$ $\{x_i(t)\}_{t\geqslant t_0}\searrow$, $\{x_\alpha(t)\}_{t\geqslant t_0}\nearrow$ where $\alpha\in V_i$ realizes the maximum at step t_0; $x_\alpha(t_0)=M_i(t_0)$.*

Proof. (a) Suppose that vertex i goes up from zero, i.e.

$$x_i(t-1)=0,\qquad x_i(t)=M_i(t-1).$$

So, there exist vertices $\alpha,\beta\in V_i$ such that

$$x_\alpha(t-1)=m_i(t-1),\qquad x_\beta(t-1)=M_i(t-1).$$

Hence, $\Delta=M_i(t-1)$. Now, if between steps $[t-1,t-2]$ vertex i is maximized then $x_\alpha(t-2)\leqslant 0$, so vertex α realizes a jump $M_i(t-1)-x_\alpha(t-2)\geqslant\Delta$. If i is minimized, $x_\beta(t-2)\geqslant 0$, so vertex β realizes a jump $x_\beta(t-2)-m_i(t-1)\geqslant -m_i(t-1)>M_i(t-1)=\Delta$. By induction one concludes the proof.

(b) Let us prove (i), the other case is similar.

If

$$x_\alpha(t_0+1)>x_\alpha(t_0) \Rightarrow x_\alpha(t_0+1)\geqslant x_i(t_0)$$
$$\Rightarrow x_\alpha(t_0+1)-x_\alpha(t_0)\geqslant x_i(t_0)-m_i(t_0)>\Delta,$$

which is a contradiction, because Δ is maximum jump. So $x_\alpha(t_0+1)\leqslant x_\alpha(t_0)$. On the other hand, one has $x_i(t_0+1)=M_i(t_0)>x_i(t_0)$.

Suppose now $t'>t_0$ is the first step where the monotonic property does not hold, i.e.

$$x_\alpha(t')\leqslant x_\alpha(t'-1)\leqslant m_i(t_0), \qquad x_i(t')\geqslant x_i(t'-1)\geqslant M_i(t_0),$$

but

$$x_\alpha(t'+1)>x_\alpha(t') \quad \text{or} \quad x_i(t'+1)<x_i(t').$$

If $x_\alpha(t'+1)>x_\alpha(t')$ holds, $x_\alpha(t'+1)\geqslant x_i(t')\geqslant M_i(t_0)$, so $x_\alpha(t'+1)-x_\alpha(t')\geqslant M_i(t_0)-m_i(t_0)>\Delta$. By applying part (a), there exists at step t_0 a jump $\Delta'\geqslant M_i(t_0)-m_i(t_0)>\Delta$, which is a contradiction.

If $x_i(t'+1)<x_i(t')$ holds, $x_i(t'+1)\leqslant x_\alpha(t')\leqslant m_i(t_0)$, so $x_i(t')-x_i(t'+1)\geqslant M_i(t_0)-m_i(t_0)>\Delta$, which concludes the proof. □

Let S_t be the set of vertices with maximum jumps between steps t and $t+1$:

$$S_t=\{i\in V\colon |x_i(t)-x_i(t+1)|\geqslant |x_s(t)-x_s(t+1)|,\ s\in\{1,\dots,n\}\}.$$

Clearly, $S_t=\emptyset$ implies $x(t)=x(t+1)$, i.e. a fixed point.

Let $H(x(t))$ be the operator

$$H(x(t))=\sum_{i\in\bigcup_{l=0}^{t}S_l}\max\{|x_i(t)-m_i(t)|,|M_i(t)-x_i(t)|\}.$$

Theorem 3.2. *For synchronous iteration on $G=(V,E)$ of NE rules*

$$x(t)\neq x(t+1) \Rightarrow H(x(t+1))>H(x(t)).$$

Proof. Clearly $S_{t+1}=S\cup\bar{S}$, such that $S\subseteq\bigcup_{l=0}^{t}S_l$ and $\bar{S}\subseteq S_{t+1}\backslash\bigcup_{l=0}^{t}S_l$, so

$$\Delta H=H(x(t+1))-H(x(t))$$
$$=\sum_{i\in\bigcup_{l=0}^{t}S_l}(\Delta H)_i+\sum_{i\in\bar{S}}\max\{|x_i(t+1)-M_i(t+1)|,|x_i(t+1)-m_i(t+1)|\},$$

where

$$(\Delta H)_i=\max\{|x_i(t+1)-M_i(t+1)|,|x_i(t+1)-m_i(t+1)|\}$$
$$-\max\{|x_i(t)-M_i(t)|,|x_i(t)-m_i(t)|\}.$$

We have to prove that $x(t) \neq x(t+1) \Rightarrow \Delta H > 0$. Clearly, if $i \in \bar{S}$, $\max\{|x_i(t+1) - M_i(t+1)|, |x_i(t+1) - m_i(t+1)|\} > 0$. To analyse $(\Delta H)_i$, for $i \in \bigcup_{l=0}^{t} S_l$, we study two cases.

Case 1. Suppose $i \in S_t$.

Case 1.1. If $x_i(t+1) = M_i(t)$, so $M_i(t) - x_i(t) < x_i(t) - m_i(t)$ hence, the ith component of $H(x(t))$ is

$$(H(x(t)))_i = x_i(t) - m_i(t),$$

since $i \in S_t \Rightarrow i \in \bigcup_{l=0}^{t+1} S_l$ and

$$\begin{aligned}(H(x(t+1)))_i &= \max\{|M_i(t+1) - x_i(t+1)|, |x_i(t+1) - m_i(t+1)|\}\\ &= \max\{|M_i(t+1) - M_i(t)|, |M_i(t) - m_i(t+1)|\}.\end{aligned}$$

Let us call $\Delta = M_i(t) - x_i(t)$ the jump of vertex i. Let $\alpha, \beta \in V_i$ vertices where the minimum and the maximum at step t are realized, i.e. $x_\alpha(t) = m_i(t)$, $x_\beta(t) = M_i(t)$. Since Δ is maximum, $x_\alpha(t+1) \leqslant m_i(t)$. In fact, suppose $x_\alpha(t+1) > m_i(t)$, α takes the maximum on its neighbor, so $x_\alpha(t+1) \geqslant x_i(t)$, hence the jump at vertex α, at step t is $x_\alpha(t+1) - m_i(t) \geqslant x_i(t) - m_i(t) > \Delta$, which is a contradiction.

We conclude $x_\alpha(t+1) \leqslant m_i(t)$ so, $m_i(t+1) \leqslant x_\alpha(t+1) \leqslant m_i(t)$, then

$$x_i(t+1) - m_i(t+1) \geqslant x_i(t+1) - m_i(t).$$

Since

$$\begin{aligned}x_i(t+1) = M_i(t) \;\Rightarrow\; M_i(t) - m_i(t+1) &\geqslant M_i(t) - m_i(t) > x_i(t) - m_i(t)\\ &= \max\{|x_i(t) - m_i(t)|, |x_i(t) - M_i(t)|\},\end{aligned}$$

so,

$$(H(x(t+1)))_i > (Hx(t))_i.$$

Case 1.2. If $x_i(t+1) = m_i(t)$, then $x_i(t) - m_i(t) < M_i(t) - x_i(t)$. Similar to the previous case, $x_\beta(t+1) \geqslant M_i(t)$. In fact if $x_\beta(t+1) < M_i(t) \Rightarrow x_\beta(t+1) \leqslant x_i(t)$, so $M_i(t) - x_\beta(t+1) \geqslant M_i(t) - x_i(t) > \Delta$, which is a contradiction. We conclude

$$\begin{aligned}(H(x(t+1))_i \geqslant M_i(t+1) - x_i(t+1) = M_i(t+1) - m_i(t) &\geqslant x_\beta(t+1) - m_i(t)\\ \geqslant M_i(t) - m_i(t) > (H(x(t))_i.&\end{aligned}$$

Case 2. Suppose $i \notin S_t$ and $i \in \bigcup_{l=0}^{t-1} S_l$, i.e. vertex i had changed by a maximum jump at step, say $t_0 \leqslant t-1$, $i \in S_{t_0}$, $i \notin \bigcup_{l<t_0} S_l$.

Case 2.1. At step t_0, vertex i jumps to the maximum. From Lemma 3.1(b), one gets $\{x_\alpha(t')\}_{t' \geqslant t_0} \searrow$ and $\{x_i(t')\}_{t' \geqslant t_0} \nearrow$. Now, at step t we know that, $m_i(t) \leqslant x_\alpha(t) \leqslant m_i(t_0)$ and $x_i(t) \geqslant M_i(t_0)$.

Furthermore, the case

$$x_i(t+1)=m_i(t) \Rightarrow M_i(t_0)=m_i(t_0),$$

which is impossible $(m_i(t_0)<M_i(t_0))$. Then, $x_i(t+1)=M_i(t)$ or $x_i(t+1)=x_i(t)$. Let us first analyse the case $x_i(t+1)=M_i(t)$. Let $\gamma\in V_i$ be the vertex which realizes the minimum at step t: $x_\gamma(t)=m_i(t)$. Clearly, $x_\gamma(t)\leqslant x_\alpha(t)$.

Suppose

$$x_\gamma(t+1)>x_\gamma(t)=m_i(t),$$

so

$$x_\gamma(t+1)\geqslant x_i(t)\geqslant M_i(t_0),$$

hence

$$x_\gamma(t+1)-x_\gamma(t)=x_\gamma(t+1)-m_i(t)\geqslant M_i(t_0)-x_\alpha(t)\geqslant \Delta'=M_i(t_0)-m_i(t_0)>\Delta$$

which implies, by Lemma 3.1(a) that at step t_0, there exists a jump $\Delta'>\Delta$, which is a contradiction. So,

$$x_\gamma(t+1)\leqslant x_\gamma(t)\leqslant m_i(t),$$

hence

$$m_i(t+1)\leqslant x_\gamma(t+1)\leqslant m_i(t)$$

and

$$\begin{aligned}x_i(t+1)-m_i(t+1)&=M_i(t)-m_i(t+1)\geqslant M_i(t)-m_i(t)\\&\geqslant \max\{|x_i(t)-m_i(t)|,\ |M_i(t)-x_i(t)|\}.\end{aligned}$$

Let us consider now the case $x_i(t+1)=x_i(t)$. As in the previous case, we have $x_i(t+1)=x_i(t)=M_i(t)$. Let $\gamma\in V_i$ be the vertex which realizes the minimum at step t, i.e. $x_\gamma(t)=m_i(t)$. Similar to the previous case, we have

$$x_\gamma(t+1)\leqslant x_\gamma(t)=m_i(t),$$

hence

$$x_i(t+1)-m_i(t+1)\geqslant M_i(t)-m_i(t)=\max\{M_i(t)-x_i(t), x_i(t)-m_i(t)\},$$

then $(\Delta H)_i\geqslant 0$.

Case 2.2. At step t_0, vertex i jumps to the minimum. From Lemma 3.1(b), one gets

$$x_\alpha(t_0)=M_i(t_0),\quad \{x_\alpha(t')\}_{t'\geqslant t_0}\nearrow \quad\text{and}\quad \{x_i(t')\}_{t'\geqslant t_0}\searrow,$$

where $\alpha\in V_i$ realizes the maximum at step t_0. We prove the result in a way similar to Case 2.1. $\square$

4. Conclusions

We have studied, from the Lyapunov operators point of view, some automata which evolves synchronously by taking local extremes as new values. Concretely we exhibit a simple class, the *Mm* automata where the Lyapunov functional permits to prove convergence to fixed points in $O(|V|^2)$ steps. Furthermore, for the NE rule, we exhibit a Lyapunov operator which permits to recuperate the convergence result establishes in [1]. In fact, since $H(x(t)) = O(n^2)$ and $\Delta H \geqslant 1$, one gets $\tau(\sigma) = O(n^2)$.

The problem to associate Lyapunov functional with automata networks is important in itself, because one may study the automata as a physical system, in this topic the evolution is a discrete process which minimizes the energy associated with the automaton. Other examples of this phenomenon and its applications can be seen in [3].

It will be interesting to determine a Lyapunov operator for the wide class of ER rules. We conjecture that there exists such an operator. Furthermore, a problem, closely related with the previous one determines whether or not there exists an automaton with exponential transient behavior. These two problems are still open. In this context we have studied in [2] a simpler update mode, the sequential iteration (i.e. one updates each node one by one in a prescribed order). For this case we exhibited a Lyapunov operator which allowed us to prove the convergence to fixed points and to determine an exponential bound for the transient time in the general case and a polynomial one for some particular rules [2].

References

[1] W. Goddard and D.J. Kleitman, Convergence of a transformation on a weighted graph, *Congr. Numer.* **82** (1991) 179–185.

[2] E. Goles and G. Hernández, Sequential iteration for extremal automata, in: E. Tirapegui et al. (eds.), *Instabilities and Nonequilibrium Structures, Vol III* (Kluwer, Dordrecht, 1993), to appear.

[3] E. Goles and S. Martínez, *Neural and Automata Networks* (Kluwer, Dordrecht, 1990).

[4] C.R. Johnson, Convergence of a non linear sharpening transformation for digital images, *SIAM J. Algebraic Discrete Methods* **6** (1985) 462–465.

[5] H.P. Kramer and J.B. Bruckner, Iterations of a nonlinear transformation for enhancement of digital images, *Pattern Recognition* **7** (1975) 53–58.